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STATISTICAL PROPERTIES OF AN ISOTROPIC RANDOM SURFACE

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CONTENTS

A number of statistical properties of a random, moving surface are obtained in the special case when the surface is Gaussian and isotropic. The results may be stated with special simplicity for a 'ring' spectrum when the energy in the spectrum is confined to one particular wavelength $\bar{\lambda}$. In particular, the average density of maxima per unit area equals $\pi/(2\sqrt{3}\,\bar{\lambda}^2)$, and the average length, per unit area, of the contour drawn at the mean level equals $\pi/(\sqrt{2}\,\bar{\lambda})$.

Introduction

Some of the statistical properties of a random, moving surface have been studied in a recent paper (Longuet-Higgins 1957)† in connexion with the analysis of sea waves. The surface was there assumed to have a correlation function of general form. In the present paper we shall discuss the special case when the surface is isotropic, that is to say, its statistical properties are independent of direction.

Although the corresponding properties of an isotropic spectrum are simpler than for a spectrum of general form, to derive them from first principles would in most cases take almost as long. In what follows, therefore, free use will be made of the more general results already obtained in (A).

The paper falls into two main sections. The first defines the parameters used to describe the surface, and discusses the relations between them. The second and main section derives various statistical properties: the distributions of elevation and gradient; the mean number of zeros along a line in arbitrary direction; the average length of contour per unit area, and the average density of maxima and minima per unit area. All these properties are independent of the motion. Next are considered the statistical distributions of the velocities of zeros, of contours and of specular points on the surface (i.e. points where the components

† This will subsequently be referred to as (A).

of the gradient take given values). The results are discussed in detail when the surface has a 'ring' spectrum, that is to say, when the energy is confined to one particular wavelength, while distributed uniformly with regard to direction.

In a final section the question is discussed of how far the spectrum is determined by its statistical properties.

1. Parameters for the surface

The energy spectrum

The surface under consideration is assumed to be representable as the sum of an infinity of long-crested waves:

$$\zeta(x,y,t) = \sum_n c_n \cos(u_n x + v_n y + \sigma_n t + \epsilon_n), \tag{1}$$

where x and y are horizontal co-ordinates and t denotes the time. The summation is over a set of wave numbers (u_n, v_n) distributed densely throughout the (u, v) plane. The frequency σ_n of each wave component depends only on the wavelength $2\pi/w_n$, where

$$w_n = (u_n^2 + v_n^2)^{\frac{1}{2}}, \tag{2}$$

and the phases ϵ_n are randomly distributed in the interval $(0, 2\pi)$. The amplitudes c_n are such that, over any element $\mathrm{d}u\,\mathrm{d}v$

$$\sum_n \tfrac{1}{2}c_n^2 = E(u,v)\,\mathrm{d}u\mathrm{d}v. \tag{3}$$

The function $E(u, v)$ is called the energy spectrum of ζ. Formally, it is the cosine transform of the correlation function $\psi(x, y)$ defined by

$$\psi(x,y) = \lim_{X,Y,T\to\infty} \frac{1}{8XYT}\int_{-X}^{X}\int_{-Y}^{Y}\int_{-T}^{T} \zeta(x',y',t')\,\zeta(x+x',y+y',t')\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t'. \tag{4}$$

In the special case considered in the present paper $E(u, v)$ is assumed to have circular symmetry about the origin, i.e.

$$E(u,v) = E(w), \tag{5}$$

say, where

$$w = (u^2+v^2)^{\frac{1}{2}}. \tag{6}$$

Moments of the spectrum

Parameters which frequently occur in the analysis of the general two-dimensional spectrum are the moments m_{pq}, m'_{pq} and m''_{pq} defined by

$$\left.\begin{aligned} m_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,u^p v^q\,\mathrm{d}u\,\mathrm{d}v,\\ m'_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,\sigma(u,v)\,u^p v^q\,\mathrm{d}u\,\mathrm{d}v,\\ m''_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,\sigma^2(u,v)\,u^p v^q\,\mathrm{d}u\,\mathrm{d}v. \end{aligned}\right\} \tag{7}$$

For example, m_{00} defines the total energy of the surface per unit area. It is assumed that the moments exist up to all orders required.

If we consider the intersection of the surface by a vertical plane in an arbitrary direction θ (that is, the plane $x \sin\theta = y \cos\theta$) the resulting curve has a one-dimensional spectrum which we may denote by $E_\theta(u')$, where u' is the wave number measured in the direction θ. The moments of this function are defined by

$$m_n(\theta) = \int_{-\infty}^{\infty} E_\theta(u')\, u'^n\, \mathrm{d}u'. \tag{8}$$

The moments $m'_n(\theta)$ and $m''_n(\theta)$ are, by definition, related to $E\sigma$ and $E\sigma^2$ in the same way that $m_n(\theta)$ is related to E. A simple relationship exists between $m_n(\theta)$ and the moments m_{pq} of the two-dimensional spectrum. On the one hand

$$m_n(\theta) = m_{n,0}\cos^n\theta + \binom{n}{1} m_{n-1,1}\cos^{n-1}\theta\sin\theta + \ldots + m_{0,n}\sin^n\theta \tag{9}$$

((A), equation (1·4·12)); on the other hand

$$m_{pq} = \frac{1}{\pi}\int_0^{2\pi} m_n(\theta)\, C_{pq}(\theta)\,\mathrm{d}\theta, \tag{10}$$

where $$C_{pq}(\theta) = \frac{1}{2\mathrm{i}^q}\left[\mathrm{e}^{\mathrm{i}n\theta} + \binom{p\quad q}{1}\bigg/\binom{n}{1}\mathrm{e}^{\mathrm{i}(n-2)\theta} + \ldots + (-1)^n\,\mathrm{e}^{-\mathrm{i}n\theta}\right] \tag{11}$$

and $\binom{p\quad q}{r}$ denotes the coefficient of x^r in the expansion of $(1+x)^p\,(1-x)^q$ (see (A), § 3·2). When the spectrum has circular symmetry, $m_n(\theta)$ is independent of θ and hence

$$m_{pq} = \begin{cases} (-1)^{\frac{1}{2}q}\binom{p\quad q}{\frac{1}{2}n}\bigg/\binom{n}{\frac{1}{2}n} m_n, & p, q \text{ both even;} \\ 0 \quad \text{otherwise.} & \end{cases} \tag{12}$$

Similar relations hold between $m'_n(\theta)$ and m'_{pq}, and between $m''_n(\theta)$ and m''_{pq}.

It is possible to describe the statistical properties of the surface in terms of the moments $m_n(\theta)$, $m'_n(\theta)$ and $m''_n(\theta)$. Nevertheless, for an isotropic spectrum it is more convenient to use the radial moments, defined by

$$M_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\, w^n\,\mathrm{d}u\,\mathrm{d}v \tag{13}$$

$$= \int_0^{\infty}\int_0^{2\pi} E(w)\, w^n w\,\mathrm{d}w\,\mathrm{d}\theta$$

$$= 2\pi\int_0^{\infty} E(w)\, w^{n+1}\,\mathrm{d}w, \tag{14}$$

and, similarly,

$$M'_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(u,v)\, E(u,v)\, w^n\,\mathrm{d}u\,\mathrm{d}v \tag{15}$$

$$= 2\pi\int_0^{\infty} \sigma(w)\, E(w)\, w^{n+1}\,\mathrm{d}w \tag{16}$$

and

$$M''_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma^2(u,v)\, E(u,v)\, w^n\,\mathrm{d}u\,\mathrm{d}v \tag{17}$$

$$= 2\pi\int_0^{\infty} \sigma^2(w)\, E(w)\, w^{n+1}\,\mathrm{d}w. \tag{18}$$

The relation of M_n to $m_n(\theta)$, when n is even, can be found as follows. We have

$$M_{2r} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,(u^2+v^2)^r\,\mathrm{d}u\,\mathrm{d}v \tag{19}$$

$$= m_{2r,0} + \binom{r}{1} m_{2r-2,2} + \binom{r}{2} m_{2r-4,4} + \ldots + m_{0,2r}$$

$$= \left[\binom{2r\quad 0}{r} - \binom{r}{1}\binom{2r-2\quad 2}{r} + \ldots (-1)^r \binom{0\quad 2r}{r}\right] m_{2r}(\theta) \Big/ \binom{2r}{r} \tag{20}$$

from equation (12). The expression in square brackets is the coefficient of x^r in

$$(1+x)^{2r} - \binom{r}{1}(1+x)^{2r-2}(1-x)^2 + \ldots + (-1)^r (1-x)^{2r} = [(1+x)^2-(1-x)^2]^r$$
$$= [4x]^r. \tag{21}$$

Hence
$$M_{2r} = 2^{2r} m_{2r}(\theta) \Big/ \binom{2r}{r} = \frac{2\,.\,4\,.\,6\,.\ldots.\,2r}{1\,.\,3\,.\,5\,.\ldots.\,(2r-1)}\, m_{2r}. \tag{22}$$

Similiarly, we have
$$M'_{2r} = \frac{2\,.\,4\,.\,6\,.\ldots.\,2r}{1\,.\,3\,.\,5\,.\ldots.\,(2r-1)}\, m'_{2r} \tag{23}$$

and
$$M''_{2r} = \frac{2\,.\,4\,.\,6\,.\ldots.\,2r}{1\,.\,3\,.\,5\,.\ldots.\,(2r-1)}\, m''_{2r}. \tag{24}$$

In particular,
$$M_0 = m_0, \quad M_2 = 2m_2, \quad M_4 = \tfrac{8}{3} m_4. \tag{25}$$

For an isotropic spectrum the odd moments vanish identically:

$$m_{2r+1}(\theta) = m'_{2r+1}(\theta) = m''_{2r+1}(\theta) = 0. \tag{26}$$

The odd moments M_{2r+1}, M'_{2r+1}, M''_{2r+1} do not occur in the present analysis.

Invariants of the spectrum

The following determinants are fundamental for the analysis of the general spectrum:

$$\Delta_0 = m_{00}, \tag{27}$$

$$\Delta_2 = \begin{vmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{vmatrix}, \tag{28}$$

$$\Delta_3 = \begin{vmatrix} m_{40} & m_{31} & m_{22} \\ m_{31} & m_{22} & m_{13} \\ m_{22} & m_{13} & m_{04} \end{vmatrix}, \tag{29}$$

and, more generally,

$$\Delta_{2r} = \begin{vmatrix} m_{2r,0} & m_{2r-1,1} & \cdots & m_{r,r} \\ m_{2r-1,1} & m_{2r-2,2} & \cdots & m_{r-1,r+1} \\ \vdots & \vdots & & \vdots \\ m_{r,r} & m_{r-1,r+1} & \cdots & m_{0,2r} \end{vmatrix}. \tag{30}$$

The vanishing of Δ_{2r} is a necessary condition for the spectrum to consist of not more than r one-dimensional spectra (see (A), §1·3).

Substitution from (12) and (25) shows that for an isotropic spectrum

$$\left.\begin{aligned} \Delta_0 &= m_0 = M_0, \\ \Delta_2 &= m_2^2 = \tfrac{1}{4}M_2^2, \\ \Delta_4 &= \tfrac{8}{27}m_4^3 = \tfrac{1}{64}M_4^3. \end{aligned}\right\} \tag{31}$$

It can be proved (see Appendix) that, for all integers $r \geqslant 0$,

$$\Delta_{2r} = \frac{1}{2^{r(r+1)}} M_{2r}^{r+1}. \tag{32}$$

As we should expect, Δ_{2r} vanishes only when M_{2r} vanishes, since an isotropic spectrum can be the sum of a finite number of one-dimensional spectra only in the trivial case when all the energy is concentrated at the origin.

Since, in an isotropic spectrum, $m_2(\theta)$ is independent of θ, we have $m_{2\,\text{max.}} = m_{2\,\text{min.}}$. Thus the long-crestedness γ^{-1} is given by

$$\gamma^{-1} = \left(\frac{m_{2\,\text{max.}}}{m_{2\,\text{min.}}}\right)^{\frac{1}{2}} = 1. \tag{33}$$

The invariant quantity $(m_{20}+m_{02})$, which is independent of the direction of the co-ordinate axes in the general case, has (from equations (12) and (25)) the value

$$m_{20}+m_{02} = M_2. \tag{34}$$

Another invariant that we shall require is the quadratic expression

$$3H = m_{40}m_{04} - 4m_{31}m_{13} + 3m_{22}^2. \tag{35}$$

Substitution from (12) gives $$3H = \tfrac{4}{3}m_4^2, \tag{36}$$

and so from (25) $$H = \tfrac{1}{16}M_4^2. \tag{37}$$

Therefore for an isotropic spectrum

$$\frac{\Delta_4^2}{H^3} = 1. \tag{38}$$

A ring spectrum

When the surface is isotropic it is impossible for the spectrum to be 'narrow' in the sense that the energy is concentrated with respect to both wavelength and direction (except in the trivial case when all the energy is at the origin). However, an interesting special case is when the energy has predominantly one wavelength $\bar{\lambda}$, that is, when it is concentrated in a narrow annular region in the (u, v) plane, with centre the origin. If $\bar{w} = 2\pi/\bar{\lambda}$ denotes the mean radius of the annulus we have approximately

$$M_n = \bar{w}^n M_0, \tag{39}$$

and hence $$M_0 M_4 = M_2^2 \tag{40}$$

or $$m_0 m_4 = \tfrac{3}{2}m_2^2. \tag{41}$$

Now from (14)

$$M_0M_4 - M_2^2 = \iiiint E(u_1, v_1)\, E(u_2, v_2)\, (w_2^4 - w_1^2 w_2^2)\, \mathrm{d}u_1\, \mathrm{d}v_1\, \mathrm{d}u_2\, \mathrm{d}v_2, \tag{42}$$

and hence

$$2(M_0M_4 - M_2^2) = \iiiint E(u_1, v_1)\, E(u_2, v_2)\, (w_1^2 - w_2^2)^2\, \mathrm{d}u_1\, \mathrm{d}v_1\, \mathrm{d}u_2\, \mathrm{d}v_2. \tag{43}$$

This quantity is always positive or zero and vanishes only when $E(u, v)$ is a ring spectrum. Further, in the isotropic case,

$$2(M_0 M_4 - M_2^2) = 4\pi^2 \int_0^\infty \int_0^\infty E(w_1)\, E(w_2)\, (w_1^2 - w_2^2)^2\, \mathrm{d}w_1\, \mathrm{d}w_2, \tag{44}$$

which, for a nearly annular spectrum, is proportional to the square of the width of the annulus. A convenient parameter for specifying the width of the annulus is therefore

$$\delta = \frac{(M_0 M_4 - M_2^2)^{\frac{1}{2}}}{M_2}. \tag{45}$$

2. Statistical properties

The distribution of surface elevation and gradient

The statistical distribution of the surface elevation ζ $(= \xi_1)$ is given by equation (2·1·8) of (A). Substituting $m_0 = M_0$ we have

$$p(\xi_1) = \frac{1}{(2\pi M_0)^{\frac{1}{2}}} \exp(-\xi_1^2/2M_0). \tag{46}$$

This is a Gaussian distribution, with mean-square value

$$\overline{\xi_1^2} = M_0. \tag{47}$$

The joint distribution of the two components of gradient

$$\frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, = \xi_1, \xi_2, \tag{48}$$

is given by

$$p(\xi_2, \xi_3) = \frac{1}{2\pi \Delta_2^{\frac{1}{2}}} \exp\left[-(m_{02}\xi_2^2 - 2m_{11}\xi_2\xi_3 + m_{20}\xi_3^2)/2\Delta_2\right] \tag{49}$$

in the general case (see (A), equation (2·1·12)). On substituting from (12) and (31) we have

$$p(\xi_2, \xi_3) = \frac{1}{\pi M_2} \exp\left[-(\xi_1^2 + \xi_2^2)/M_2\right], \tag{50}$$

a symmetrical Gaussian distribution in two dimensions. The distributions of ξ_1 and (ξ_2, ξ_3) are statistically independent (see (A), § 2·1).

Let us write

$$(\xi_2, \xi_3) = (\alpha \cos\theta, \alpha \sin\theta), \quad \frac{\partial(\xi_2, \xi_3)}{\partial(\alpha, \theta)} = \alpha \tag{51}$$

in (50), so that α and θ denote the magnitude and direction of the surface gradient. Then we have for the joint distribution of α and θ

$$p(\alpha, \theta) = \frac{\alpha}{\pi M_2} \exp(-\alpha^2/M_2), \tag{52}$$

which is of course independent of θ. The mean-square slope of the surface is given by

$$\overline{\alpha^2} = M_2. \tag{53}$$

The distribution of the slope α, regardless of θ, is a Rayleigh distribution:

$$p(\alpha) = \frac{2\alpha}{M_2} \exp(-\alpha^2/M_2). \tag{54}$$

The distribution of θ, regardless of α, is a constant:

$$p(\theta) = \frac{1}{2\pi}. \tag{55}$$

The number of zeros per unit distance

If we consider the curve of intersection of the surface by a vertical plane in the direction θ, we may count the number N_0 of zeros of this curve per unit horizontal distance. From (A), equation (2·2·5), N_0 is given by

$$N_0 = \frac{1}{\pi}\left(\frac{m_2(\theta)}{m_0(\theta)}\right)^{\frac{1}{2}} = \frac{1}{\pi}\left(\frac{M_2}{2M_0}\right)^{\frac{1}{2}}. \tag{56}$$

Similarly from equation (2·2·10) of (A) the number of maxima and minima of the curve per unit distance is given by

$$N_1 = \frac{1}{\pi}\left(\frac{m_4(\theta)}{m_2(\theta)}\right)^{\frac{1}{2}} = \frac{1}{\pi}\left(\frac{3M_4}{4M_2}\right)^{\frac{1}{2}}. \tag{57}$$

In general, the number of zeros, per unit distance, of the rth derivative of the curve is given by

$$N_r = \frac{1}{\pi}\left(\frac{m_{2r+2}(\theta)}{m_{2r}(\theta)}\right)^{\frac{1}{2}} = \frac{1}{\pi}\left(\frac{2r+1}{2r+2}\,\frac{M_{2r+2}}{M_{2r}}\right)^{\frac{1}{2}}. \tag{58}$$

Also from (2·2·12) and (2·2·13) of (A) the number of points per unit distance where the curve crosses the arbitrary level $\zeta = \xi_1$ is

$$N_0(\xi_1) = \frac{1}{\pi}\left(\frac{M_2}{2M_0}\right)^{\frac{1}{2}} \exp(-\xi_1^2/2M_0), \tag{59}$$

and the number of times when the gradient of the curve takes the arbitrary value ξ_2 is

$$N_0(\xi_2) = \frac{1}{\pi}\left(\frac{3M_4}{4M_2}\right)^{\frac{1}{2}} \exp(-\xi_2^2/M_2). \tag{60}$$

For a ring spectrum, we have

$$N_0 = \frac{\overline{w}}{\pi}\left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{2}{\overline{\lambda}}\left(\frac{1}{2}\right)^{\frac{1}{2}}, \tag{61}$$

$$N_1 = \frac{\overline{w}}{\pi}\left(\frac{3}{4}\right)^{\frac{1}{2}} = \frac{2}{\overline{\lambda}}\left(\frac{3}{4}\right)^{\frac{1}{2}}, \tag{62}$$

and, in general,

$$N_r = \frac{\overline{w}}{\pi}\left(\frac{2r+1}{2r+2}\right)^{\frac{1}{2}} = \frac{2}{\overline{\lambda}}\left(\frac{2r+1}{2r+2}\right)^{\frac{1}{2}}, \tag{63}$$

where

$$\overline{\lambda} = \frac{2\pi}{\overline{w}} \tag{64}$$

denotes the characteristic wavelength of the spectrum. For a long-crested wave of the same wavelength, the number of zero crossings per unit distance would be $2/\overline{\lambda}$ in a direction at right angles to the crests, and zero in a direction parallel to the crests. Equation (63) shows that, for an isotropic spectrum, N_r is always less than the maximum value $2/\overline{\lambda}$. On the other hand, for large values of r, N_r approaches this value.

The length and direction of contours

Let contours be drawn on the surface at the level $\zeta = \xi_1 =$ constant; the length of contour lying in any given horizontal area will be, on the average, proportional to that area. The mean length $\bar{s}$ per unit area is shown in (A), § 2·3, to be given by

$$\bar{s}(\xi_1) = \frac{1}{\pi}\left(\frac{m_{20}+m_{02}}{m_{00}}\right)^{\frac{1}{2}} \frac{\mathrm{E}(\sqrt{(1-\gamma^2)})}{\sqrt{(1+\gamma^2)}} \exp\left(-\xi_1^2/2m_0\right), \tag{65}$$

where γ^{-1} denotes the long-crestedness and $\mathrm{E}(k)$ is the Legendre elliptic integral of the first kind. For an isotropic spectrum we have

$$\gamma = 1, \quad \mathrm{E}(\sqrt{(1-\gamma^2)}) = \tfrac{1}{2}\pi, \tag{66}$$

and hence
$$\bar{s}(\xi_1) = \frac{1}{2\sqrt{2}}\left(\frac{M_2}{M_0}\right)^{\frac{1}{2}} \exp\left(-\xi_1^2/2M_0\right). \tag{67}$$

The distribution of contour direction for an isotropic spectrum is of course uniform.

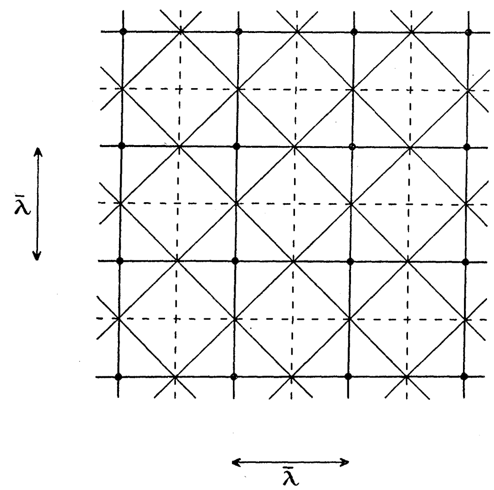

FIGURE 1. The pattern formed by two systems of regular waves intersecting at right angles. ——, crests of the individual wave systems; – – – –, troughs of the individual wave systems; ——, contours of mean level in the combined system; ●, maxima of the combined system.

For a ring spectrum the average length of contour becomes

$$\bar{s}(\xi_1) = \frac{\bar{w}}{2\sqrt{2}} \exp\left(-\xi_1^2/2M_0\right), \tag{68}$$

and in particular at the mean level $\zeta = 0$ we have

$$\bar{s}(0) = \frac{\bar{w}}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}\frac{1}{\bar{\lambda}} = 2\cdot 22\ldots\frac{1}{\bar{\lambda}}. \tag{69}$$

This result may be compared with the simple pattern made by two regular sine waves of equal wavelength $\bar{\lambda}$ and of equal amplitude intersecting at right angles (see figure 1). It is easy to see that the contours of zero level run diagonally, making angles of $\frac{1}{4}\pi$ with the directions of the two sine waves. The distance between adjacent parallel contours is $\bar{\lambda}/\sqrt{2}$. The mean length of contour for each diagonal direction is therefore $\sqrt{2}/\bar{\lambda}$, and the total mean length is twice this, or

$$\bar{s}(0) = 2\sqrt{2}\frac{1}{\bar{\lambda}} = 2\cdot 83\ldots\frac{1}{\bar{\lambda}}. \tag{70}$$

This is somewhat greater than in the isotropic case. However, $\bar{s}(\xi_1)$ falls off in a different fashion in the two cases; clearly for the two intersecting waves $\bar{s}(\xi_1)$ vanishes when ξ_1 exceeds twice the amplitude of each wave.

The density of maxima and minima

A very interesting problem is that of the density of maxima, minima or stationary points per unit horizontal area. It is shown in (A) that for any statistically uniform surface the density of maxima $D_{ma.}$ together with the density of minima $D_{mi.}$ equals the density of saddle points $D_{sa.}$. Also that for the special surface represented by equation (1), $D_{ma.} = D_{mi.}$. It follows that

$$D_{ma.} = D_{mi.} = \tfrac{1}{4}D_{sta.}, \qquad D_{sa.} = \tfrac{1}{2}D_{sta.}, \tag{71}$$

where $D_{sta.}$ denotes the total density of stationary points. The density of specular points, that is, points where the two components of gradient take given values ξ_2, ξ_3 is given by

$$D_{sp.} = D_{sta.}[-(\xi_2^2+\xi_3^2)/M_2] \tag{72}$$

for an isotropic surface (cf. (A), equation (2·4·66)).

The evaluation of $D_{ma.}$ in terms of the energy spectrum of the surface is given by

$$D_{ma.} = \frac{1}{2\pi^2}\frac{l_1}{\Delta_2^{\frac{1}{2}}}\Phi(-l_2/l_1), \tag{73}$$

where $l_1 \geqslant 0 \geqslant l_2 \geqslant l_3$, these being the three roots (always real) of the cubic equation

$$4l^3 - 3Hl - \Delta_4 = 0, \tag{74}$$

and where Φ is a function involving complete elliptic integrals (see (A), equation (2·4·53)). Substituting for Δ_4 and H from (31) and (48) we have

$$64l^3 - 3M_4 l - M_4^3 = 0, \tag{75}$$

and so

$$(l_1, l_2, l_3) = (\tfrac{1}{4}M_4, -\tfrac{1}{8}M_4, -\tfrac{1}{8}M_4). \tag{76}$$

Since, from equation (2·5·55) of (A),

$$\Phi(\tfrac{1}{2}) = \frac{\pi}{2\sqrt{3}}, \tag{77}$$

we have

$$D_{ma.} = \frac{1}{2\pi^2}\frac{M_4}{2M_2}\frac{\pi}{2\sqrt{3}} = \frac{1}{8\sqrt{3}\,\pi}\frac{M_4}{M_2}. \tag{78}$$

In particular, for a ring spectrum,

$$D_{ma.} = \frac{1}{8\sqrt{3}\,\pi}\bar{w}^2 = \frac{\pi}{2\sqrt{3}}\frac{1}{\bar{\lambda}^2} = 0{\cdot}907\ldots\frac{1}{\bar{\lambda}^2}. \tag{79}$$

This may be compared with the corresponding result for two regular sine waves of equal wavelength $\bar{\lambda}$, and equal amplitude, crossing at right angles; in that case

$$D_{ma.} = \frac{1}{\bar{\lambda}^2} \tag{80}$$

simply. So the density of maxima in the isotropic case is slightly less than for the two intersecting sine waves.

The velocities of zeros

If a plane section of the surface be taken in a direction θ as before, we may consider the distribution of the velocities of points on this curve which lie at a given level, say $\zeta = \xi_1$. This would be equivalent to drawing a contour map of the surface and finding the velocities of the intersections of the contours $\zeta = \xi_1$ with a fixed line in direction θ.

For a general spectrum, the distribution of the velocity is given by

$$p(c)_{\xi_1} = \frac{1}{2}\frac{\Delta_{2'}/m_2^{\frac{1}{2}}}{(m_0''+2m_1'c+m_2c^2)^{\frac{3}{2}}} \tag{81}$$

(see (A), equation (2·5·14)), where

$$\Delta_{2'} = m_0''m_2 - m_1'^2. \tag{82}$$

For an isotropic spectrum we have

$$\left.\begin{aligned} m_0 &= M_0, & m_1' &= 0, \\ m_0'' &= M_0'', & m_2 &= \tfrac{1}{2}M_2, \end{aligned}\right\} \tag{83}$$

giving

$$p(c)_{\xi_1} = \frac{M_0''/M_2}{(c^2+2M_0''/M_2)^{\frac{3}{2}}}. \tag{84}$$

This distribution is symmetrical about the origin, as we should expect. Its second moment and standard deviation are infinite, but a measure of its width is the interquartile range, given by

$$2\sqrt{\left(\frac{2}{3}\right)}\left(\frac{M_0''}{M_2}\right)^{\frac{1}{2}} \tag{85}$$

For a ring spectrum this becomes

$$2\sqrt{\left(\frac{2}{3}\right)}\frac{\bar{\sigma}}{\bar{w}} = 2\sqrt{(\tfrac{2}{3})}\,\bar{c}, \tag{86}$$

where $\bar{c}$ is the phase velocity of the component waves.

It will be noticed that the distribution of c is independent of the particular contour $\zeta = \xi_1$ at which the velocity is measured.

Similarly, we may consider the velocities c_1 of the maxima and minima of the curve. From equation (2·5·19) of (A) we find for the distribution of c_1

$$p(c_1) = \frac{2M_2''/3M_4}{(c^2+4M_2''/3M_4)^{\frac{3}{2}}}. \tag{87}$$

This is of the same form as $p(c)$ but with an interquartile range of width

$$\frac{4}{3}\left(\frac{M_2''}{M_4}\right)^{\frac{1}{2}} \tag{88}$$

For a ring spectrum this becomes

$$\frac{4}{3}\frac{\bar{\sigma}}{\bar{w}} = \tfrac{4}{3}\bar{c} \tag{89}$$

The distributions of the velocities of higher derivatives of the curve may be found in a similar way.

The motion of the contours

The motion of a contour may be defined as follows. Let P be a fixed point through which the contour passes at a given time, and let straight lines be drawn through P parallel to the axes Ox, Oy. The intersections of the contour with these two lines will move with velocities c_x, c_y, say, which determine completely the local motion of the contour. If any other fixed line is drawn through P in a direction θ, and if c is the speed of the contour intersection along it, then it can be shown that

$$\frac{1}{c} = \frac{1}{c_x}\cos\theta + \frac{1}{c_y}\sin\theta. \tag{90}$$

The reciprocals $1/c_x$, $1/c_y$ will be denoted by κ_x, κ_y respectively.

Alternatively we may consider the components q_x, q_y of the velocity of the contour *normal to itself* at P. Between (κ_x, κ_y) and (q_x, q_y) there is a reciprocal relationship:

$$\left.\begin{aligned}(q_x, q_y) &= \left(\frac{\kappa_x}{\kappa_x^2+\kappa_y^2}, \frac{\kappa_y}{\kappa_x^2+\kappa_y^2}\right),\\ (\kappa_x, \kappa_y) &= \left(\frac{q_x}{q_x^2+q_y^2}, \frac{q_y}{q_x^2+q_y^2}\right)\end{aligned}\right\} \tag{91}$$

(see (A), § 2·6).

The statistical distribution of (κ_x, κ_y) is given by equation (2·6·21) of (A). In the general case,

$$p(\kappa_x, \kappa_y)_{\xi_1} = \frac{1}{\pi\Delta_3^{\frac{1}{2}}(m_{20}+m_{02})^{\frac{1}{2}}}\frac{\sqrt{(1+\gamma^2)}}{\mathrm{E}\,(\sqrt{(1-\gamma^2)})}\frac{(\kappa_x^2+\kappa_y^2)^{\frac{1}{2}}}{R}, \tag{92}$$

where

$$\Delta_3 = \begin{vmatrix} m_{20} & m_{11} & m'_{10} \\ m_{11} & m_{02} & m'_{01} \\ m'_{10} & m'_{01} & m''_{00}\end{vmatrix}, \tag{93}$$

γ^{-1} is the long-crestedness, E is the Legendre elliptic integral of the first kind, and

$$R = M_{11}\kappa_x^2 + 2M_{12}\kappa_x\kappa_y + M_{22}\kappa_y^2 - 2M_{13}\kappa_x - 2M_{23}\kappa_y + M_{33}, \tag{94}$$

in which (M_{ij}) is the matrix inverse to that of Δ_3. In the isotropic case we have

$$\Delta_3 = \begin{vmatrix} \frac{1}{2}M_2 & 0 & 0 \\ 0 & \frac{1}{2}M_2 & 0 \\ 0 & 0 & M''_0\end{vmatrix}, \tag{95}$$

and so

$$R = \frac{2}{M_2}(\kappa_x^2+\kappa_y^2) + \frac{1}{M''_0}. \tag{96}$$

Setting also $\gamma = 1$ in (92) we have

$$p(\kappa_x, \kappa_y)_{\xi_1} = \frac{2}{\pi^2}\left(\frac{M_2}{2M''_0}\right)^{\frac{1}{2}}\frac{(\kappa_x^2+\kappa_y^2)^{\frac{1}{2}}}{[(\kappa_x^2+\kappa_y^2)+M_2/2M''_0]^2}, \tag{97}$$

whence also

$$p(q_x, q_y)_{\xi_1} = \frac{2}{\pi^2}\left(\frac{2M''_0}{M_2}\right)^{\frac{3}{2}}\frac{(q_x^2+q_y^2)^{-\frac{1}{2}}}{[(q_x^2+q_y^2)+2M''_0/M_2]^2}. \tag{98}$$

These are symmetrical distributions, independent of direction in the horizontal plane. If we write

$$(q_x, q_y) = (q\cos\theta, q\sin\theta), \tag{99}$$

so that q is the absolute value of the normal velocity, we have

$$p(q)_{\xi 1} = 2\pi p(q,\theta)_{\xi 1} = 2\pi q p(q_x, q_y)_{\xi 1}, \tag{100}$$

or from (98)

$$p(q)_{\xi 1} = \frac{4}{\pi}\left(\frac{2M_0''}{M_2}\right)^{\frac{3}{2}} \frac{1}{(q^2+2M_0''/M_2^2)^2}. \tag{101}$$

This distribution has a mode at the origin $q = 0$. The mean-square value of the velocity is given by

$$\overline{q^2} = \frac{2M_0''}{M_2}. \tag{102}$$

For a ring spectrum,

$$\overline{q^2} = \frac{2\overline{\sigma}^2}{\overline{w}^2} = 2\overline{c}^2, \tag{103}$$

where $\overline{c}$ is the velocity of a component sine wave in the spectrum.

The velocities of specular points

As defined above, a specular point is a point that would be seen by a distant observer as a point of reflexion of a distant source of light. We may imagine such a point to be followed continuously. If its velocity is denoted by (c_x, c_y) then the distribution of (c_x, c_y) is shown in (A), § 2·7, to depend upon the matrix

$$(\Xi_{ij}) = \left(\begin{array}{ccc|cc} m_{40} & m_{31} & m_{22} & m'_{30} & m'_{21} \\ m_{31} & m_{22} & m_{13} & m'_{21} & m'_{12} \\ m_{22} & m_{13} & m_{04} & m'_{12} & m'_{03} \\ \hline m'_{30} & m'_{21} & m'_{12} & m''_{20} & m''_{11} \\ m'_{21} & m'_{12} & m'_{03} & m''_{11} & m''_{02} \end{array}\right) \tag{104}$$

If (M_{ij}) denotes the inverse of this matrix, then it is found that

$$p(c_x, c_y)_{\xi_1,\xi_2} = \frac{1}{16\pi^2(\Delta_2\Delta_5)^{\frac{1}{2}} D_{\text{ma.}}} \frac{3(n_{13}-n_{22})^2+N(N_{22}-4N_{31})}{N^{\frac{5}{2}}}, \tag{105}$$

where (N_{ij}) is the symmetric 3×3 matrix whose components are

$$\left.\begin{aligned} N_{11} &= M_{44}c_x^2 - 2M_{41}c_x + M_{11}, \\ N_{22} &= M_{55}c_x^2 + 2M_{45}c_xc_y + M_{44}c_y^2 - 2M_{52}c_x - 2M_{42}c_y + M_{22}, \\ N_{33} &= M_{55}c_y^2 - 2M_{53}c_y + M_{33}, \\ N_{23} &= M_{55}c_xc_y + M_{45}c_y^2 - M_{53}c_x - (M_{43}+M_{52})\,c_y + M_{23}, \\ N_{31} &= M_{45}c_xc_y - M_{43}c_x - M_{51}c_y + M_{31}, \\ N_{12} &= M_{45}c_x^2 + M_{44}c_xc_y - (M_{42}+M_{51})\,c_x - M_{41}c_y + M_{12}, \end{aligned}\right\} \tag{106}$$

and where

$$\Delta_2 = \begin{vmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{vmatrix}, \qquad \Delta_5 = |\,\Xi_{ij}\,|, \qquad N = |\,N_{ij}\,|, \tag{107}$$

$$\left.\begin{aligned} n_{13} &= N_{21}N_{32} - N_{22}N_{31}, \\ n_{22} &= N_{11}N_{33} - N_{13}^2. \end{aligned}\right\} \tag{108}$$

($D_{\text{ma.}}$ denotes the density of maxima.) For an isotropic spectrum these expressions are considerably simplified. Thus (Ξ_{ij}) becomes

$$(\Xi_{ij}) = \left(\begin{array}{ccc|cc} \frac{3}{8}M_4 & 0 & \frac{1}{8}M_4 & 0 & 0 \\ 0 & \frac{1}{8}M_4 & 0 & 0 & 0 \\ \frac{1}{8}M_4 & 0 & \frac{3}{8}M_4 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2}M_2'' & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}M_2'' \end{array}\right), \tag{109}$$

and so

$$(M_{ij}) = \left(\begin{array}{ccc|cc} \frac{3}{M_4} & 0 & -\frac{1}{M_4} & 0 & 0 \\ 0 & \frac{8}{M_4} & 0 & 0 & 0 \\ -\frac{1}{M_4} & 0 & \frac{3}{M_4} & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{2}{M_2''} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{M_2''} \end{array}\right). \tag{110}$$

Hence we have

$$(N_{ij}) = \frac{1}{M_4}\begin{pmatrix} 2\xi^2+3 & 2\xi\eta & -1 \\ 2\xi\eta & 2(\xi^2+\eta^2)+8 & 2\xi\eta \\ -1 & 2\xi\eta & 2\eta^2+3 \end{pmatrix}, \tag{111}$$

where
$$(\xi, \eta) = \left(\frac{M_4}{M_2''}\right)^{\frac{1}{2}} (c_x, c_y). \tag{112}$$

After some reduction we find from (105)

$$p(c_x, c_y)_{\xi_1, \xi_2} = \frac{4\sqrt{3}}{\pi}\frac{M_4}{M_2''}\frac{(\alpha^2+4)(3\alpha^2+4)(\alpha^2+6)+6\alpha^4}{[(\alpha^2+4)(3\alpha^2+4)]^{\frac{5}{2}}}, \tag{113}$$

where
$$\alpha^2 = \xi^2+\eta^2 = \frac{M_4}{M_2''}(c_x^2+c_y^2). \tag{114}$$

To find the distribution of the non-dimensional velocity α we may write

$$(\xi, \eta) = (\alpha\cos\theta, \alpha\sin\theta), \tag{115}$$

and so
$$p(\alpha) = 2\pi\alpha p(\xi, \eta) = 2\pi\alpha\frac{M_2''}{M_4}p(c_x, c_y), \tag{116}$$

giving
$$p(\alpha) = 8\sqrt{3}\,\alpha\frac{(\alpha^2+4)(3\alpha^2+4)(\alpha^2+6)+6\alpha^4}{[(\alpha^2+4)(3\alpha^2+4)]^{\frac{5}{2}}}. \tag{117}$$

The form of the distribution is shown in figure 2. There is a single maximum, at $\alpha = 0{\cdot}72$ approximately. At infinity $p(\alpha)$ is $O(\alpha^{-3})$, so that the second moment diverges. The median value (dividing the distribution into two equal parts) is at

$$\alpha = 1{\cdot}240\ldots = \alpha_m, \tag{118}$$

say. The value α_m has the following significance: if the positions of the specular points are noted at two successive instants t and $t+\mathrm{d}t$, half of the points may be observed to have moved through distances greater than

$$\left(\frac{M_2''}{M_4}\right)^{\frac{1}{2}} \alpha_m \,\mathrm{d}t \tag{119}$$

from their original positions.

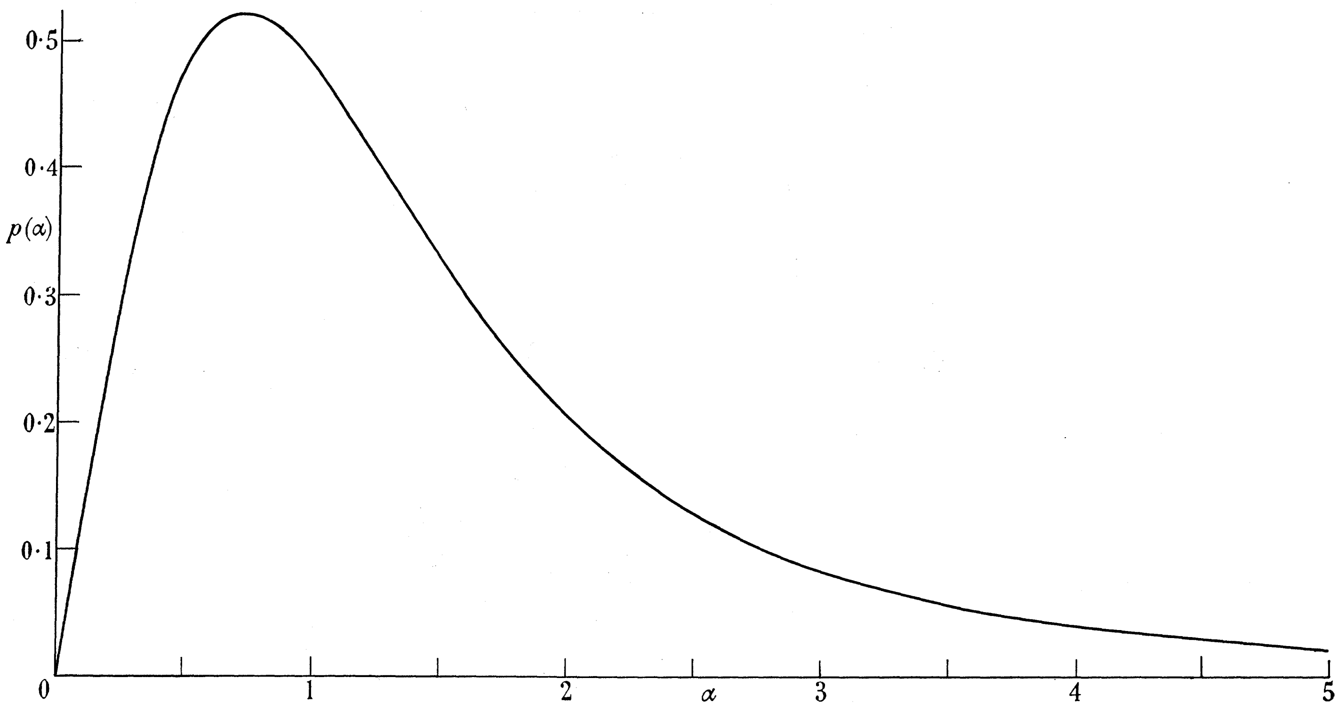

FIGURE 2. Graph of $p(\alpha)$, showing the form of the probability distribution of the absolute velocities of specular points.

For a ring spectrum this median distance is

$$\alpha_m \bar{c}\,\mathrm{d}t, \tag{120}$$

where $\bar{c}$ is the velocity corresponding to the wave number $\overline{w}$.

3. On the uniqueness of the spectrum

Suppose we are given certain of the statistical properties discussed in § 2, the question arises whether these determine the spectrum uniquely, or to what extent the properties may be shared by other spectra.

The correlation function $\psi(x,y)$, if known for all values of x and y, would suffice to determine $E(u,v)$ under general conditions; for E is simply the cosine transform of ψ. However, the properties discussed above are purely local, that is to say, they involve the behaviour of the surface at one point and its immediate neighbourhood. We have seen that these properties depend only on the moments M_{2r}, M''_{2r} of the spectrum (which are the derivatives of ψ at the origin). In fact M_{2r} is the rth moment with respect to β $(=w^2)$, of the function

$$F(\beta) = F(w^2) = \pi E(w) \tag{121}$$

defined over $0<\beta<\infty$.

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The properties which depend on moments M_{2r} up to order $r = s$ will be said to be of order $2s$. If all the properties of order up to $2s$ are known, the moments up to order $2s$ may all be determined. For example, from (47) and (59) we have

$$\left.\begin{aligned} M_0 &= \overline{\zeta^2}, \\ \frac{M_{2r+2}}{M_{2r}} &= \frac{2r+2}{2r+1}\pi^2 N_r^2 \quad (r = 0, 1, 2, \ldots), \end{aligned}\right\} \tag{122}$$

and therefore
$$M_{2r} = \frac{2.4.6.\ldots.2r}{1.3.5.\ldots.(2r-1)}\pi^{2r}\overline{\zeta^2}N_0^2 N_1^2 \ldots N_{r-1}^2. \tag{123}$$

Suppose first that the moments of F exist and are known up to infinite order. This does not determine F uniquely in general (see Kendall 1952, chap. 4), but if certain restrictions are placed upon F for large values of β—for example, if E is exponentially small at infinity—then only one function with these moments can exist.

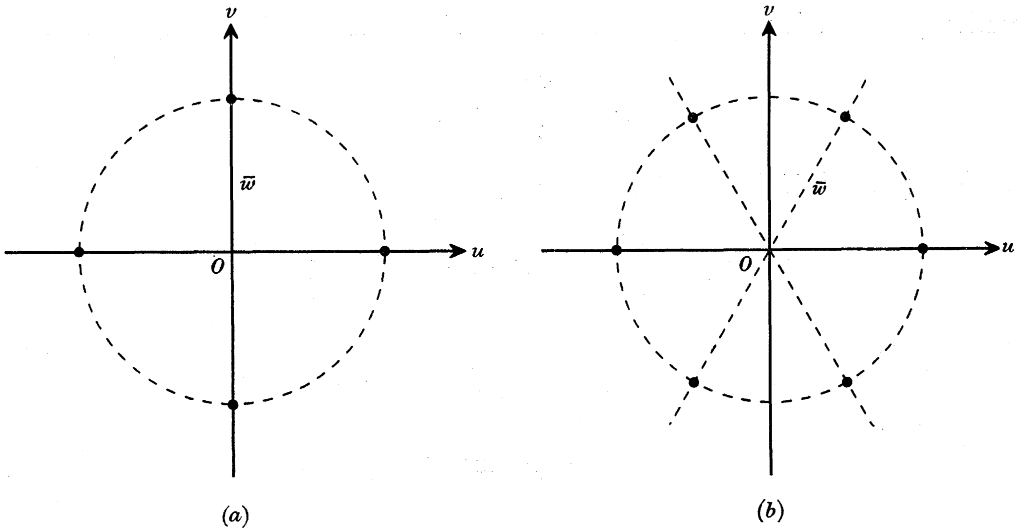

FIGURE 3. Examples of spectra whose statistical properties are isotropic (*a*) to order 2, (*b*) to order 4.

In practice we may know, or be concerned with, properties up to a finite order $2s$ only. Except in special cases, if the moments are known up to a finite order only, infinitely many functions may be found having these same moments. A particularly simple function is one consisting of the sum of delta functions. It can be shown (Stieltjes 1894, chap. 1) that if the moments M_{2r} are specified for $r = 0, 1, \ldots, s$, then a function F^*, the sum of not more than $[\frac{1}{2}s+1]$ positive delta functions, may be found having these same moments and lying in the range $0 \leqslant \beta < \infty$. Hence a combination of not more than $[\frac{1}{2}s+1]$ ring spectra may be found which has the required statistical properties, to order $2s$.

One can also find non-isotropic spectra with the same statistical properties. Consider, for example, a surface which is the sum of two pairs of long-crested, incoherent systems of waves, of equal wavelength and mean-square amplitude, intersecting at right angles (figure 3*a*). The spectrum function has the same moments m_{00}, m_{20}, m_{11}, m_{02} as an isotropic ring spectrum of the same radius $\overline{w}$, for if

$$m_{00} = M_0, \tag{124}$$

then
$$(m_{20}, m_{11}, m_{02}) = (\tfrac{1}{2}\overline{w}^2 M_0, \quad 0, \quad \tfrac{1}{2}\overline{w}^2 M_0). \tag{125}$$

Now by equation (9) both $m_0(\theta)$ and $m_2(\theta)$ depend only on these moments; thus they are the same as for a ring spectrum, and so independent of θ. Hence the number N_0 of zeros per unit distance, which is given by equation (56), is also independent of the direction θ. Similarly, all properties depending only on moments of order 0 and 2 will appear as isotropic, including the distributions of surface slopes and of contour direction.

More generally, if we consider a surface which is the sum of $s+1$ pairs of long-crested waves travelling in directions $\theta = j\pi/(s+1)$ uniformly spaced between 0 and 2π, then all the moments m_{pq} of order $p+q \leqslant 2s$ are the same as for a ring spectrum. Hence N_0, N_1, ..., N_{s-1} are all independent of θ, and so are all properties of order less than or equal to $2s$.

The case $s = 2$ is shown in figure 3*b*. For this surface both the number N_0 of zeros and the number N_1 of maxima and minima in a direction θ are independent of θ.

The general theorem may be quite simply proved as follows. Consider any spectrum in which the energy is all concentrated on the circle $w = \overline{w}$, and in which the distribution of energy with regard to θ is given by some function $G(\theta)$. The (p, q)th moment of the spectrum is then

$$m_{pq} = \int_0^{2\pi} u^p v^q G(\theta)\,d\theta, \tag{126}$$

where $(u, v) = (\overline{w}\cos\theta, \overline{w}\sin\theta)$. That is to say,

$$m_{pq} = \int_0^{2\pi} \overline{w}^{p+q} \cos^p\theta \sin^q\theta\, G(\theta)\,d\theta. \tag{127}$$

The product $\cos^p\theta \sin^q\theta$ can be expressed as a trigonometric series in θ containing terms in $\cos n\theta$ and $\sin n\theta$, where n does not exceed $p+q$. Suppose then that the Fourier series for $G(\theta)$ is of the form

$$G(\theta) = \overline{G} + \sum_{n=1}^{\infty} [A_n \cos n\theta + B_n \sin n\theta], \tag{128}$$

where

$$\left.\begin{aligned} A_1 = A_2 = \ldots = A_{2s+1} = 0, \\ B_1 = B_2 = \ldots = B_{2s+1} = 0. \end{aligned}\right\} \tag{129}$$

Then if $p+q \leqslant (2s+1)$ all terms in (126) will vanish except those arising from the constant term $\overline{G}$. This gives

$$m_{pq} = \int_0^{2\pi} \overline{w}^{p+q} \cos^p\theta \sin^q\theta\, \overline{G}\,d\theta. \tag{130}$$

In other words, m_{pq} is the same as for a ring spectrum. Now when the spectrum consists of s uniformly spaced pairs of wave systems we have

$$G(\theta) = \frac{\pi\overline{G}}{s+1} \sum_{j=1}^{2s+2} \delta\left(\theta - \frac{j\pi}{s+1}\right), \tag{131}$$

where $\delta(\theta)$ denotes the Dirac delta function. So

$$A_n = \frac{1}{\pi}\int_0^{2\pi} G(\theta)\cos n\theta\,d\theta = \frac{\overline{G}}{s+1}\sum_{j=1}^{2s+2} \cos\frac{jn\pi}{s+1}, \tag{132}$$

which vanishes when $n = 1, 2, \ldots, (2s+1)$; and similarly for B_n. Thus the conditions (128) are satisfied.

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As a corollary it follows that any spectrum which is periodic in θ with period $\pi/(s+1)$ will have isotropic properties up to order $2s+1$. For the spectrum may be considered as the sum of regular systems of point spectra of the type just discussed.

Appendix

Proof of equation (32)

Substitution from (7) into (30) gives

$$\Delta_{2r} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1, v_1)\ldots E(u_{r+1}, v_{r+1})$$
$$\times \begin{vmatrix} u_1^{2r} & u_2^{2r-1}v_2 & \ldots & u_{r+1}^{r}v_{r+1}^{r} \\ u_1^{2r-1}v_1 & u_2^{2r-2}v_2^2 & \ldots & u_{r+1}^{r-1}v_{r+1}^{r+1} \\ \vdots & \vdots & & \vdots \\ u_1^r v_1^r & u_2^{r-1}v_2^{r+1} & \ldots & v_{r+1}^{2r} \end{vmatrix} \mathrm{d}u_1\,\mathrm{d}v_1\ldots\mathrm{d}u_{r+1}\,\mathrm{d}v_{r+1} \quad \text{(A 1)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1, v_1)\ldots E(u_{r+1}, v_{r+1})$$
$$\times \begin{vmatrix} 1 & 1 & \ldots & 1 \\ v_1/u_1 & v_2/u_2 & \ldots & (v_{r+1}/u_{r+1}) \\ \vdots & \vdots & & \vdots \\ (v_1/u_1)^r & (v_2/u_2)^r & \ldots & (v_{r+1}/u_{r+1})^r \end{vmatrix}$$
$$\times (u_1/v_1)^r\,(u_2/v_2)^{r-1}\ldots(u_{r+1}/v_{r+1})^0$$
$$\times (u_1v_1\,.\,u_2v_2\,.\ldots.\,u_{r+1}v_{r+1})^r\,\mathrm{d}u_1\,\mathrm{d}v_1\ldots\mathrm{d}u_{r+1}\,\mathrm{d}v_{r+1}. \quad \text{(A 2)}$$

The value of Δ_{2r} is unaltered by permuting the suffixes $1, 2, \ldots, (r+1)$ among themselves. Thus, adding all the $(r+1)!$ different permutations we have

$$(r+1)!\,\Delta_{2r} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1 v_1)\ldots E(u_{r+1}, v_{r+1})$$
$$\times \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ (v_1/u_1)^r & \ldots & (v_{r+1}/u_{r+1})^r \end{vmatrix} \times \begin{vmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ (u_1/v_1)^r & \ldots & (u_{r+1}/v_{r+1})^r \end{vmatrix}$$
$$\times (u_1v_1u_2v_2\ldots u_{r+1}v_{r+1})^r\,\mathrm{d}u_1\,\mathrm{d}v_1\ldots\mathrm{d}u_{r+1}\,\mathrm{d}v_{r+1}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1 v_1)\ldots E(u_{r+1}, v_{r+1})$$
$$\times \prod_{p>q}(u_p/v_p - u_q/v_q)\prod_{p>q}(v_p/u_p - v_q/u_q)$$
$$\times (u_1v_1\ldots u_{r+1}v_{r+1})^r\,\mathrm{d}u_1\,\mathrm{d}v_1\ldots\mathrm{d}u_{r+1}\,\mathrm{d}v_{r+1}. \quad \text{(A 3)}$$

Now writing

$$\left.\begin{aligned} (u_p, v_p) &= (w_p\cos\theta_p,\ w_p\sin\theta_p), \\ E(u_p, v_q) &= E(w_p), \end{aligned}\right\} \quad \text{(A 4)}$$

in the above, we have

$$(r+1)!\,\Delta_{2r} = \int_0^\infty \int_0^{2\pi} \dots \int_0^\infty \int_0^{2\pi} E(w_1) \dots E(w_{r+1}) \times \prod_{p>q} \sin^2(\theta_p - \theta_q)\, w_1^{r+1} \dots w_{r+1}^{r+1}\, dw_1\, d\theta_1 \dots dw_{r+1}\, d\theta_{r+1}$$

$$= \left(\frac{M_{2r}}{2\pi}\right)^{r+1} \int_0^{2\pi} \dots \int_0^{2\pi} \prod_{p>q} \sin^2(\theta_p - \theta_q)\, d\theta_1 \dots d\theta_{r+1}. \qquad \text{(A 5)}$$

The multiple integral may be evaluated as follows. Since

$$\sin^2(\theta_p - \theta_q) = \tfrac{1}{4}(e^{2i\theta_p} - e^{2i\theta_q})(e^{-2i\theta_p} - e^{-2i\theta_q}), \qquad \text{(A 6)}$$

we have

$$\prod_{p>q} \sin^2(\theta_p - \theta_q) = \frac{1}{2^{r(r+1)}} \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{2i\theta_1} & e^{2i\theta_2} & \dots & e^{2i\theta_{r+1}} \\ \vdots & \vdots & & \vdots \\ e^{2ri\theta_1} & e^{2ri\theta_2} & \dots & e^{2ri\theta_{r+1}} \end{vmatrix} \times \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{-2i\theta_1} & e^{-2i\theta_2} & \dots & e^{-2i\theta_{r+1}} \\ \vdots & \vdots & & \vdots \\ e^{-2ri\theta_1} & e^{-2ri\theta_2} & \dots & e^{-2ri\theta_{r+1}} \end{vmatrix}. \qquad \text{(A 7)}$$

A typical term in the expansion of the first determinant is

$$1 \,.\, e^{2i\theta_2} \,.\, e^{4i\theta_3} \,.\,\dots\,.\, e^{2ri\theta_{r+1}}, \qquad \text{(A 8)}$$

which, when multiplied into the second determinant, gives

$$\begin{vmatrix} 1 & e^{2i\theta_2} & \dots & e^{2ri\theta_{r+1}} \\ e^{-2i\theta_1} & 1 & \dots & e^{2(r-1)i\theta_{r+1}} \\ \vdots & \vdots & & \vdots \\ e^{-2ri\theta_1} & e^{-2(r-1)i\theta_2} & \dots & 1 \end{vmatrix}. \qquad \text{(A 9)}$$

The integral of this determinant over the given ranges of $\theta_1, \dots, \theta_{r+1}$ is

$$\begin{vmatrix} 2\pi & 0 & \dots & 0 \\ 0 & 2\pi & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 2\pi \end{vmatrix} = (2\pi)^{r+1}. \qquad \text{(A 10)}$$

Since the first determinant in (A 7) contributes altogether $(n+1)!$ such terms we have

$$\int_0^{2\pi} \dots \int_0^{2\pi} \prod_{p>q} \sin^2(\theta_p - \theta_q)\, d\theta_1 \dots d\theta_{r+1} = \frac{(r+1)!\,(2\pi)^{r+1}}{2^{r(r+1)}}. \qquad \text{(A 11)}$$

From this and (A 5) the result follows.

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